Applied Modelling and Simulation of Technological Systems
P. Borne and S.G. Tzafestas (Editors)
Elsevier Science Publishers B.V. (North-Holland)

LINEAR STATE ESTIMATION IN THE PRESENCE OF SUDDEN SYSTEM CHANGES-AN EXPERT SYSTEM

ANASTASIOS POULIEZOS
POLYTECHNIC OF CRETE
Eleft. Venizelou 34
CHANIA-CRETE-GREECE

G.S. STAVRAKAKIS
c/o Prof. S.TZAFESTAS
Control and Automation Group
Electrical Engineering Dept.
NATIONAL TECHNICAL UNIVERSITY
ZOGRAFOU-ATHENS 15773, GREECE

ABSTRACT

The class of stochastic linear systems that are subject to additive changes of unknown magnitude in the state variables or in the system parameters, occuring at unknown times, is considered. At first, a classification of the changes which may occur in a system is given. Following this the performance of the discrete Kalman-Bucy filter when the system is subjected to sudden changes modelled by any additive function, is evaluated. The results concerning the effect of the various kinds of faults on the Kalman-Bucy filter innovations are summarised in two useful tables.

1. INTRODUCTION

In applying the discrete linear Kalman filter to a real system, the model parameter matrices, the noise variances and the filter initialisations must be specified a priori. For such problems the Kalman-Bucy filter performs extremely well. However, if the Kalman filter is operating with correctly identified parameters, then a sudden change in the real system will introduce errors in the model parameters, which unless taken into account, will produce degradation of filter performance, i.e. an increase in the state estimate error variance or a bias in the state estimate. A *system change* is any change of known source in the assumed system parameters, which occurs at known time with known magnitude. This sudden change causes a degradation in the performance of the state estimation procedure.
If the source of the system change or the time that the change occurs or the magnitude of the change are unknown then the system change is defined as a *system fault*.
The design of a fault monitoring scheme consists of various functions to be performed in the event of a fault.
The first stage, in order to proceed with the fault monitoring functions is to model the various faults and to derive the system model, when the system is subjected to any of these faults. In this paper we consider the various fault models and the effect of faults on the properties of the residual sequence.

In the case of *system change* these results can be used to reorganize the system, which entails reinitialisation of model and filter parameters. The reader is refered to [1], [2], [3] for the problem of detection of jumps in linear systems.

2. MODELS OF ADDITIVE FAULTS OR CHANGES

The general form of a fault model will be assummed *additive*, i.e. given a system parameter p(k) and a fault modelled by h(k,θ,v) where k is the sample time, θ is the time of fault occurrence and v is the size of the fault, then the value of the parameter after a fault will be given by

$$P_{new}=P_{old}+h(k,\theta,v)$$

where $v \in (V_\ell, V_u)$

is the size of the fault constrained below by V_ℓ and above by V_u, and $\theta \in (0,\infty)$ is the time of fault occurrence which takes a finite integer value if a fault occurs and is infinite otherwise.
Faults may be classified into three types:
type I: jump, *type II*: step, *type III*: ramp and higher order. Type I faults may modelled by the term

$$V\delta_{k,\theta} \text{ where } \delta_{k,\theta} = 1;\ k=\theta \quad \text{(kronecker delta)}$$
$$\qquad = 0 \ \ k\neq\theta$$

If a fault has not occurred θ is infinite, hence $\delta_{k,\infty}=0$. This model may used for instantaneous faults of one time unit duration. Type II faults may be modelled by the term

$$V\sigma_{k,\theta} \text{ where } \sigma_{k,\theta}=1;\ k\geq\theta \quad =0;\ k<\theta$$

This model may be used for faults of constant size which have a permanent effect on the system. Type III faults may be modelled by the term $h(v,k)\sigma_{k,\theta}$ where $h(v,k)$ is a polynomial in k; v may used to represent faults of changing magnitude. A ramp could then be represented as $(a+kv)\sigma_{k,\theta}$

Such models introduce further complexity to the fault monitoring scheme. However, ramps could be approximated by a series of steps. This approach will depend on the slope of the ramp, which should not be too steep for such approximation to be valid.

3. MODELLING OF SYSTEMS SUBJECT TO TYPE II FAULTS

Consider the following discrete-time dynamical system:

$$x(k+1)=\phi(k+1,k)x(k)+w(k) \tag{1}$$

$$y(k)=H(k)x(k)+v(k) \tag{2}$$

where $x(k)\varepsilon R^n$ is the state with Gaussian initial condition $x(0)$ of mean $\hat{x}_0$ and covariance P_0. In addition $y\varepsilon R^p$ is the observation, and $\{w(k)\}$, $\{v(k)\}$ are independent, zero mean, white Gaussian sequences with $E(w(k)w(k)^T)=Q(k)$ and $E(v(k)v(k)^T)=R(k)$.

Faults of type I will have a temporary effect on the system performance since the Kalman filter will resettle at its pre-fault condition. Assume θ=k+1. If the filter was started at time k+1, the effect of the initial error at time θ would diminish, under stability assumptions, as successive measurements are processed. However, the time taken for the Kalman filter to settle will in general be greater than the estimation delay time t_e and therefore in applications where accuracy in state estimation is vital at every time, a fault monitoring scheme should be employed for faults of this type 3 .

The following models are proposed in the case of faults in the state equation:

a. Step bias in plant state

$$x(k+1)=\phi(k+1,k)x(k)+w(k)+v_x\sigma_{k+1,\theta} \tag{3}$$

Step changes in the mean of the plant noise sequence w(k) may also be modelled in this manner.

b. Step change in φ matrix

$$x(k+1)=(\phi(k+1,k)+\Delta\phi\sigma_{k+1,\theta})x(k)+w(k) \tag{4}$$

c. Additional plant noise.

$$x(k+1)=\phi(k+1,k)x(k)+w(k)+\zeta_x(k)\sigma_{k+1,\theta} \tag{5}$$

where $\zeta_x(k)$ is conveniently defined as a white gaussian random sequence, independent of $x(0)$, $w(i)$, $v(i)$ for all i,k and of zero mean and unknown constant variace S_x. In the above cases the observation equation remains as in (2). The following models are proposed in the case of faults in the measurement equation:

d. Step bias in the measurements

$$y(k)=H(k)x(k)+v(k)+v_y\sigma_{k,\theta} \tag{8}$$

Step changes in the value of the measurement noise sequence v(k) may also modelled in this manner.

e. Step change in H matrix

$$y(k)=(H(k)+\Delta H\sigma_{k,\theta})x(k)+v(k) \tag{9}$$

f. Additional measurement noise

$$y(k)=H(k)x(k)+v(k)+\zeta_y(k)\sigma_{k,\theta} \tag{10}$$

where $\zeta_y(k)$ is conveniently defined as a gaussian sequence of zero mean and unknown constant variance S_y independent of $x(0)$, $w(i)$, $v(i)$ for all i,k. The models developed here may be used in situations where a fault may occur *in only one parameter at any given instant.* Such faults may be called *single* faults. The same approach can howver be extended to the modelling of simultaneous occurrence of faults in more than one parameter, termed *multiple* faults.

4. EFFECT OF FAULTS ON KALMAN FILTER RESIDUALS

The innovations sequence $\{\gamma(k)\}$ of the discrete Kalman-Bucy filter is given by the well known equation:

$$\gamma(k)=y(k)-H(k)\hat{x}(k/k-1) \tag{11}$$

where $\hat{x}(k/k-1)$ is the optimum estimate of x(k) based on the measurement sequence:

$$Y^{k-1}=\{y(i)\}\quad i=1\ldots k-1 \tag{12}$$

The matrix P(k/k-1) is the variance of the estimation error $x(k)-\hat{x}(k/k-1)$;K(k is the Kalman-Bucy filter gain. Given the observability conditions, the true system is observable through the measurement sequence $\{y(k)\}$ only.

The well known equations of the Kalman Bucy filter [4] imply that knowing the measurement residual sequence $\{\gamma(k)\}$ i equivalent to knowing $\{y(k)\}$.

It is shown in [4] that for a Kalman filter operating with correctly identified parameters, the residual sequence $\{\gamma(k)\}$ $k=1,2,\ldots$ is gaussian with:

$$\gamma(k)=E\{\gamma(k)\}=0;\ \text{all } k$$
$$C(k,m)=E(\gamma(k)\gamma^T(m))=0;\ \text{all } k\neq m \quad (12^*)$$
$$C(k,k)=E\{\gamma(k)\gamma^T(k)\}=H(k)P(k/k-1)H^T(k)+R(k)$$

It therefore follows that $\{\gamma(k)\}$ will contain information of faults provided that the faults are observable. The following general theorem describes the effect of additive faults on the innovation sequence of the discrete Kalman-Bucy filter.

4.1. Theorem

The state, measurement, filter state estimate, inovations sequence and covariance matrix for models represented by equations (1)-(2) which are subject to sudden faults modelled by any additive function may be expressed as:

$$x(k)=x_0(k)+h_x(k,\theta,\Delta P) \quad (13)$$
$$y(k)=y_0(k)+h_y(k,\theta,\Delta P) \quad (14)$$
$$\hat{x}(k/k)=\hat{x}_0(k/k)+f(k,\theta,\Delta P) \quad (15)$$
$$\gamma(k)=\gamma_0(k)+g(k,\theta,\Delta P) \quad (16)$$
$$P(k/k)=P_0(k/k)+P_f(k,\theta,\Delta P) \quad (17)$$

where,

$h_x(k,\theta,\Delta P)$ is the effect on state $x(k)$ of a fault of size ΔP, which occured at time θ,

$h_y(k,\theta,\Delta P)$ is the corresponding effect on measurement $y(k)$,

$f(k,\theta,\Delta P)$ is the effect on the state estimate $\hat{x}(k/k)$,

$g(k,\theta,\Delta P)$ is the effect on the innovations $\gamma(k)$

$P_f(k,\theta,\Delta P)$ is the effect on the covariance matrix $P(k/k)$ and $x_0(k/k)$, $y_0(k/k)$, $\hat{x}_0(k/k)$, $\gamma_0(k)$, $P_0(k/k)$ represent the values of the corresponding variables that would be obtained if no fault occurs.

Further, the recursions on h_x, h_y, f, g and P_f are given by:

$$g(k,\theta,\Delta P)=h_y(k,\theta,\Delta P)-H(k)\phi(k,k-1)f(k-1,\theta,\Delta P) \quad (18)$$
$$f(k,\theta,\Delta P)=K(k)g(k,\theta,\Delta P)+\phi(k,k-1)f(k-1,\theta,\Delta P);\ k\geq\theta \quad (19)$$
$$P_f(k,\theta,\Delta P)=(h_x(k,\theta,\Delta P)-f(k,\theta,\Delta P))(h_x(k,\theta,\Delta P)-f(k,\theta,\Delta P))^T;\ k\geq\theta \quad (20)$$
$$g(k,\theta,\Delta P)=f(k,\theta,\Delta P)=0;\ k<\theta \quad (21)$$

The proof is given in Appendinx I.

The quantities h_x and h_y depend on the particular fault but in view of (2) if a fault occurs in a parameter of the plant equation:

$h_x(k,\theta,\Delta P)\neq 0$; modelled as in the previous section and

$h_y(k,\theta,\Delta P)=H(k)h_x(k,\theta,\Delta P)$; $k\geq\theta$

If a fault occurs in a parameter of the measurement equation,

$$h_x(k,\theta,\Delta P)=0;\ \text{all } k$$
$$h_y(k,\theta,\Delta P)\neq 0;\ k\geq\theta$$

If a fault does not occur, since θ is infinite, h_x and h_y are identically zero. Equations (13)-(21) provide a model for the evolution of the $\{x(k)\}$, $\{y(k)\}$, $\{\hat{x}(k/k)\}$ $\{\gamma(k)\}$ and $\{P(k/k)\}$.

4.2. Innovations modelling subject to type II faults.

For type I faults we refer the reader to [3]. For type II faults appropriate h_x and h_y functions can be calculated using the models developed in the previous section.

Corollary: Since the effects of faults are additive, type II faults may be thought of as a series of successive type I faults. In this way, the effect of a type II fault of size ΔP can be found by considering the total effect of successive type I faults of equal, but unknown, size ΔP.

a. Step bias in plant state

The effect of conescutive jumps starting at time θ up to and inclunding time $k\,(k\geq\theta)$ will therefore be:

$$g_a(k,\theta)v_x+g_a(k,\theta+1)v_x+\ldots+g_a(k,k)v_x$$

Therefore, $h_x(k,\theta,v_x)=v_x(\theta)$; $k\geq\theta$
$=0$ $\quad k<\theta$

Applying the results of the previous theorem the residual sequence may then be written as:

$$\gamma(k)=\gamma_0(k)+\sum_{i=\theta}^{k} g_a(k,i)v_x(i);\ k\geq\theta$$

where,

$$g_a(k,\theta)=H(k)(\phi(k,\theta)-\phi(k,k-1)f_a(k-1,\theta))$$
$$f_a(k,\theta)=K(k)g_a(k,\theta)+\phi(k,k-1)f_a(k-1,\theta);\quad k\geq\theta \quad (22)$$
$$g_a(k,\theta)=f_a(k,\theta)=0;\ k<\theta$$

b. State change in ϕ

Consider equation (4) when $\theta=k+1$:

$$x(k+1)=(\phi(k+1,k)+\Delta\phi)x(k)+w(k)$$
$$=x_0(k+1)+\Delta\phi x_0(k)$$

At time k+2, the following equationshold:

$$x(k+2)=(\phi(k+2,k+1)+\Delta\phi)x(k+1)+w(k+1)$$
$$=(\phi(k+2,k+1)+\Delta\phi)(x_0(k+1)+\Delta\phi x_0(k))+w(k+1)$$

$$=x_0(k+2)+\Delta\phi x_0(k+1)+\phi(k+2,k+1)\Delta\phi x_0(k)+$$
$$+\Delta\phi^2 x_0(k)$$

It follows by inspection that the desired expression for the effect of a step change in ϕ is given by:

$$h_x(k,\theta,\Delta\phi)=\sum_{i=\theta}^{k}\sum_{q=0}^{\eta=k-i} C_\eta^q\phi(i+\eta,i+q)\Delta\phi^q\Delta\phi x_0(i-1);\quad k\geq\theta \tag{23}$$

where $C_\eta^q=\dfrac{\eta!}{q!(\eta-q)!}$, $C_\eta^0=C_\eta^\eta=1;\ 0<q<\eta$

In this case, the effect of a step change cannot be written as the sum of effects of successive jump changes.
However, this problem can be examined if someone defines:

$$h_x^*(i,j,\Delta\phi)=\sum_{q=0}^{\eta=i-j} C_\eta^q\phi(j+\eta,j+q)\Delta\phi^q\Delta\phi x_0(j-1) \tag{24}$$

Then, $h_x(k,\theta,\Delta\phi)=\sum_{i=\theta}^{k} h_x^*(k,i,\Delta\phi)$ and the terms of the sum can be thought of as an effect of a jump fault.

The results of the previous theorem can be then applied directly and the filter residuals may be expressed as:

$$\gamma(k)=\gamma_0(k)+\sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_0(i-1)$$

where g_b and f_b are recursively computed from:

$$g_b(k,i,\Delta\phi)=H(k)\Big(\sum_{q=0}^{\eta=k-i} C_\eta^q\Phi(i+\eta,i+q)\Delta\phi^q-$$
$$-\Phi(k,k-1)f_b(k-1,i,\Delta\phi)\Big)$$

$$f_b(k,i,\Delta\phi)=K(k)g_b(k,i,\Delta\phi)+\Phi(k,k-1)$$
$$f_b(k-1,i,\Delta P);\quad k\geq i \tag{25}$$
$$g_b(k,i,\Delta\phi)=f_b(k,i,\Delta\phi)=0;\quad k<i$$

It may be seen that in the case of a step change in Φ, the unknown size of the fault is nonlinearly related to the additional terms g_b.
In the cases of additional plant noise, step change in H, step bias in measurements and additional measurement noise the cumulative property holds, so the appropriate functions can be calculated similarly to the case of a step bias in plant state.
In fact, it is easy to see that, in the case of additional plant noise,

$$h_x(k,\theta,\zeta_x)=\sum_{i=\theta}^{k}\Phi(k,i)\zeta_x(i) \tag{26}$$

in the case of step change in H,

$$h_y(k,\theta,\Delta H)=\Delta Hx_0(\theta);\quad k=\theta$$
$$=0\quad;\quad k\neq\theta \tag{27}$$

in the case of step bias in measurements

$$h_y(k,\theta,v_y)=v_y(\theta)\quad;\quad k=\theta$$
$$=0\quad;\quad k\neq\theta \tag{28}$$

and in the case of additional measurement noise,

$$h_y(k,\theta,\zeta_y)=\zeta_y(\theta)\quad;\quad k=\theta$$
$$=0\quad;\quad k\neq\theta$$

The results concerning the effect of the various kinds of type II faults on the form of the Kalman-Bucy filter residuals are summarised in the Table 1 below.

TABLE 1.

	bias	additional noise	change in transition or measurement matrix
Plant	$\gamma(k)=\gamma_0(k)+\sum_{i=\theta}^{k} g_a(k,i)v_x(i)$	$\gamma(k)=\gamma_0(k)+\sum_{i=\theta}^{k} g_c(k,i)\zeta_x(i)$	$\gamma(k)=\gamma_0(k)+\sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_0(i-1)$
	g_a, g_c are both calculated by : $g(k,\theta)=H(k)[\Phi(k,\theta)-\Phi(k,k-1)f(k-1,\theta)]$ $f(k,\theta)=K(k)g(k,\theta)+\Phi(k,k-1)f(k-1,\theta)\ ;\ k\geq\theta$ $g(k,\theta)=f(k,\theta)=0\ ;\ k<\theta$		$g_b(k,i,\Delta\phi)$ is recursively computed from equations (25)
Measurements	$\gamma(k)=\gamma_0(k)+\sum_{i=\theta}^{k} g_d(k,i)v_y(i)$	$\gamma(k)=\gamma_0(k)+\sum_{i=\theta}^{k} g_f(k,i)\zeta_y(i)$	$\gamma(k)=\gamma_0(k)+\sum_{i=\theta}^{k} g_e(k,i)\Delta Hx_0(i)$
		g_d, g_e, g_f are all calculated from : $g(k,\theta)=-H(k)\Phi(k,k-1)f(k-1,\theta)\ ;\ k\geq\theta$ $f(k,\theta)=K(k)g(k,\theta)+\Phi(k,k-1)f(k-1,\theta)\ ;\ k\geq\theta$ $g(k,k)=1$ $g(k,\theta)=f(k,\theta)=0\ ;\ k<\theta$	

5. EFFECT OF TYPE II FAULTS ON THE JOINT P.D.F OF THE INNOVATIONS

Having established the form of the innovations sequence under faulty conditions, their joint probability distribution (JOINT P.D.F) will now be examined.
In normal operation, the statistical properties of the residuals are given by (12-*). When different cases of type II fault occur, the residuals generated by the Kalman filter evolve according to TABLE 1.
For every possible fault, it is necessary to establish which of the statistical properties of the residuals in normal operation remain the same, and which are subject to change.
Since the linear tructure of the Kalman filter equations and state and measurement models is not changed in the presence of an additive type fault, the residuals remain a linear combination of the gaussian measurement sequence $\{y(k)\}$ and are therefore also gaussian. This result means that the joint pdf of the innovations will be completely characterised by its first and second moments.
To ease notational complexity the following definitions are made:

$$\gamma^{j,k^T} = \left[\gamma(j)^T \gamma(j+1)^T \ldots \gamma(k)^T\right] \in R^{n^*},$$

$$\bar{\gamma}^{j,k} = E[\gamma^{j,k}],$$

where $n^* = n(k-j+1)$ and

$$C^{j,k} = \text{cov}\gamma^{j,k} = E\left[(\gamma^{j,k} - \bar{\gamma}^{j,k})(\gamma^{j,k} - \bar{\gamma}^{j,k})^T\right] \in R^{n^* \times n^*}$$

Using these definitions the pdf of the gaussian vector $\gamma^{j,k}$ is:

$$P(\gamma^{j,k}) = \frac{1}{(2\pi)^{n^*/2}|C^{j,k}|^{1/2}} \exp\{-\frac{1}{2}(\gamma^{j,k} - \bar{\gamma}^{j,k})^T (C^{j,k})^{-1} (\gamma^{j,k} - \bar{\gamma}^{j,k})\} \quad (29)$$

The effect of the faults on the whiteness property must be examined as well. If a fault has not occured (29) becomes:

$$P(\gamma^{j,k}) = \prod_{m=j}^{k} \frac{1}{(2\pi)^{n/2} |C(m,m)|^{1/2}} \exp\{-\frac{1}{2}(\gamma_0(m)^T C(m,m)^{-1} \gamma_0(m)\} \stackrel{\Delta}{=} \pi(j,k) \quad (30)$$

Since $E\{\gamma_0(m)\} = 0$ and $C^{j,k} = \text{diag}\left[C(m,m)\right]$,

$$m = j, j+1, \ldots, k$$

a. Joint p.d.f of residuals in the event of step bias in plant state.

The mean of the residuals sequence in this case is given as:

$$E[\gamma(k)] = E\left[\gamma_0(k) + \sum_{i=\theta}^{k} g_a(k,i) V_x(i)\right]$$

and since the residuals in normal operation $\gamma_0(k)$ have zero mean and the second term in the expectation is nonrandom,

$$E\left[\gamma(k)\right] = \sum_{i=\theta}^{k} g_a(k,i) V_x \stackrel{\Delta}{=} \bar{\gamma}(k)$$

Therefore the residual mean vector is:

$$\bar{\gamma}_a^{j,k^T} = \left[0, 0, \ldots, (g_a(\theta,\theta)V_x)^T, \ldots, (\sum_{i=\theta}^{k} g_a(k,i)V_x)^T\right]; \theta \geq j \quad (31)$$

The residual covariance matrix can be calculated considering,

$$E\left[(\gamma(k) - \bar{\gamma}(k))(\gamma(m) - \bar{\gamma}(m))^T\right]$$

But, $\gamma(k) - \bar{\gamma}(k) = \gamma_0(k) + \sum_{i=\theta}^{k} g_a(k,i)V_x - \sum_{i=\theta}^{k} g_a(k,i)V_x = \gamma_0(k)$

Hence,

$$\text{cov}\left[\gamma(k)\gamma(m)^T\right] = 0 \;;\; k \neq m$$
$$= C(k,k);\; k = m \quad (32)$$

This result implies that a step bias in the plant state does not change the correlation properties of the innovations sequence. The joint p.d.f. of the residual sequence may be written as:

$$P(\gamma^{j,k}) = \pi(j,\theta-1) \prod_{i=\theta}^{k} \frac{1}{(2\pi)^{n/2} |C(i,i)|^{1/2}} \exp\{-\frac{1}{2}((\gamma(i) - \bar{\gamma}(i))^T C(i,i)^{-1} (\gamma(i) - \bar{\gamma}(i))\}$$

(33)

b. Joint p.d.f of residuals in the event of a step change in ϕ.

The mean of the residual sequence in this case is given as:

$$E\,\gamma(k) = E\,\gamma_0(k) + \sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_0(i-1)$$
$$= \sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi E\, x_0(i-1) \quad (34)$$

Using (1),

$$E[x_0(i-1)] = \phi(i-1,0)\bar{x}(0)$$

Under system stability assumptions, in system steady state, $E\, x_0(i) = 0$; all $i >> 0$. This result implies that if g_b remains bounded for all k, the mean value of the residual sequence in the event of a change in ϕ is zero, if the fault occurs when the system has reached steady state. Under stability conditions for $\phi(k+1,k)+\Delta\phi$, see 4 , that is if the change does not destabilize the system, g_b will remain bounded. Under these circumstances the residual covariance is

$$cov\left[\gamma(k),\gamma(m)\right] = E\left[\gamma(k)\gamma^T(m)\right]$$

$$= E\Big[(\gamma_0(k)+\sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_0(i-1))\,(\gamma_0(m)+\sum_{j=\theta}^{m} g_b(m,j,\Delta\phi)\Delta\phi x_0(j-1))^T\Big]$$

or

$$cov\left[\gamma(k),\gamma(m)\right] = C(k,m)+\sum_{j=\theta}^{m} E\left[\gamma_0(k)x_0(j-1)^T\right]\Delta\phi^T g_b^T(m,j,\Delta\phi)+\sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi E\left[x_0(i-1)\gamma_0(m)^T\right]+E\{\sum_{i=\theta}^{k} g_b(k,i,\Delta\phi)\Delta\phi x_0(i-1)\cdot\sum_{j=\theta}^{m} x_0(j-1)^T\Delta\phi^T g_b^T(m,j,\Delta\phi)\} \quad (35)$$

Since $\gamma_0(k)$ is independent of $x_0(i)$, $i=0,1,\ldots k-1$, assuming $0<<k<m$ without loss of generality, the third sum in (35) vanish. The stochastic process $x(k)$ has the following property, from (1):

$$cov\{x(k)x(m)^T\}=\phi(m,k)\{var\ x(k)\};\ m>k.$$

Hence, the second sum in (35) becomes:

$$\sum_{j=k+1}^{m} E\{\gamma_0(k)x_0(j-1)^T\}\Delta\phi^T g_b^T(m,j,\Delta\phi),\ \text{Where}$$

$$E\{\gamma_0(k)x_0^T(j-1)\}=\phi(j-1,k)\cdot H\cdot P(k/k-1)$$

The last sum is given by the expression:

$$g_b(k,\theta,\Delta\phi)\Delta\phi\sum_{j=\theta}^{m} E\{x_0(\theta-1)x_0(j-1)^T\}\Delta\phi^T g_b^T(m,j,\Delta\phi)+g_b(k,\theta+1,\Delta\phi)\Delta\phi\sum_{j=\theta}^{m} E\{x_0(\theta)x_0(j-1)^T\}\Delta\phi^T g_b^T(m,j,\Delta\phi)+\ldots\ldots\ldots.+g_b(k,k,\Delta\phi)\Delta\phi\sum_{j=\theta}^{m} E\{x_0(k-1)x_0(j-1)^T\}\Delta\phi^T g_b^T(m,j,\Delta\phi).$$

c. Joint p.d.f of residuals in the event of additional plant noise.

The expected value of $\gamma(k)$ is zero in this case, since both $\gamma_0(k)$ and $\xi_x(i)$ are of zero mean.

$$cov\{\gamma(k)\gamma(m)\}=C(k,m)+\sum_{i=\theta}^{\lambda} g_c(k,i)g_c(m,i)^T S_x$$

$$\lambda=\min\{k,m\} \quad (36)$$

Since,

$$E\left[\xi_x(j)v(i)^T\right]=0;\ \text{all } i,j,\quad E\left[\xi_x(i)\xi_x^T(j)\right]=0;\ \text{all } i\neq j$$

and the $\xi_x(i)$ is independent of the $\gamma_0(i)$

The residual covariance matrix is then given by:

$$C_c^{j,k}=\begin{bmatrix} C^{j,\theta-1} & 0 \\ 0 & C_c^{\theta,k}\end{bmatrix},\ \text{where}$$

$$C_c^{\theta,k}=\begin{bmatrix} C(\theta,\theta)+C'_c(\theta,\theta) & C'_c(\theta+1,\theta)\ldots\ldots & C'_c(\theta,k) \\ C'_c(\theta+1,\theta) & C(\theta+1,\theta+1)+C'_c(\theta+1,\theta+1)\ldots. & C'_c(\theta+1,k) \\ \vdots & & \vdots \\ C'_c(\theta,k) & \ldots & C(k,k)+C'_c(k,k)\end{bmatrix}$$

$$\text{and } C'_c(i,j)=\sum_{m=\theta}^{\lambda} g_c(i,m)g_c(j,m)^T S_x \quad (38)$$

It can be seen that the residual sequence following an increase in the plant noise is not stationary as well as not white, since in general, $C_c(i,j)\neq C_c(i+m,j+m)$

d. Joint p.d.f of residuals in the event of step bias in the measurements.

The expected value and covariance of the innovations sequence are found in the same way as in the case of a step bias in the state. Thus,

$$\bar{\gamma}_d^{j,k\,T}=\left[0,0\ldots,(g_d(\theta,\theta)v_y)^T,\ldots,(\sum_{i=\theta}^{k} g_d(k,i)v_y)^T\right],\ \theta\geq j \quad (39)$$

Since the step bias is non-random the correlation properties do not change,

$$cov\left[\gamma(k),\gamma(m)\right]=E\left[\gamma_0(k)\gamma_0(m)^T\right]=C(k,m) \quad (40)$$

These results imply that joint pdf can be written as a product of independent random variables as in (33).

e. Joint p.d.f of residuals in the event of a step change in H.

Since the $x_0(i)$ are random and E is a linear operator,

$$E\left[\gamma(k)\right]=\sum_{i=\theta}^{k} g_e(k,i)\Delta HE\left[x_0(i)\right] \quad (41)$$

Hence, under steady state conditions, $E\ x_0(i)=0$, $i>>0$. Thus,

$$E\left[\gamma(k)\right]=0$$

The covariance is calculated as,

$$E\left[\gamma(k)\gamma(m)\right]=C(k,m)+\sum_{j=\theta}^{m} E\left[\gamma_0(k)x_0(j)\right]^T\Delta H^T g_e(m,j)+\sum_{i=\theta}^{k} g_e(k,i)\Delta HE\left[x_0(i)\gamma_0(m)^T\right]+E\{\sum_{i=\theta}^{k} g_e(k,i)\Delta Hx_0(i)\cdot\sum_{j=\theta}^{m} x_0(j)^T\Delta H^T g_e^T(m,j)\} \quad (42)$$

The individual sums are of the same form as the sums involved in the covariance function of the residuals following a step change in ϕ, so they can be calculated using the same considerations.

f. Joint p.d.f of residuals in the event of additional measurement noise.

Since the $J_y(i)$ have zero mean,

$$E\left[\gamma(k)\right]=0 \quad (43)$$

The covariance is given by:

$$\text{cov}\left[\gamma(k)\gamma(m)\right]=C(k,m)+\sum_{i=\theta}^{\lambda} g_f(k,i)\,g_f^T(m,i)\,S_y \quad (44)$$

where $\lambda=\min\{k,m\}$, since $\mathcal{J}_y$ is independent of $v(i)$ for all i, $\mathcal{J}_y(i)$ is independent of $\mathcal{J}_y(j)$ for all $i\neq j$ and $\mathcal{J}_y(i)$ is independent of $\gamma(i)$.

This results implies that an increase in the measurement noise has the same qualitative effect on the joint pdf of the innovations sequence as the increase in plant noise, i.e the residuals remain zero mean but become correlated and nonstationary.

6. SYMMARY OF RESULTS AND COMMENTS

The effects of the type II faults are summarised in Table 2. As it can be seen from Table 2, if the faults occur in steady state, then they may be classified into two disjoint classes, as follows:

C_1: {faults with effect of nonzero mean of residuals}

C_2: {faults with effect of correlated residuals}

or, equivalently,

C_1: {faults a, d}

C_2: {faults b, c, e, f}

The no-fault class,

C_0: {no fault}

may also be added, so that the three classes fully characterise any probable condition of the system.

The results concerning the stationarity property of the residuals, in steady state following a fault are quite important. This property, together with the fact that the correlation decreases exponentially, ensures that time averages are meaningful 5 . Thus, even under faulty conditions, the sequence of residual values can be considered to be ensemble values of the corresponding distributions and hence on-line fault monitoring is possible.

A fourth class could also be included, covering cases outside the main assumptions of the problem. This would include situations where a fault in the transition coefficients occurs when the system is in the transient state or situations in which $|\phi+\Delta\phi|>1$, i.e. the change in ϕ destabilises the system. The common feature of the effect on the innovations sequence of faults of this class, is the introduction of bias as well as correlation. Therefore, C_3 may be defined as:

C_3: {faults with effect of nonzero mean and correlation}

or equivalently,

C_3: {(b or e in transient state) or (destabilising b)}

TABLE 2: Effect of faults on innovations, in steady state conditions of system and filter.

	zero mean	independence	stationarity
a. State bias	no	yes	yes
b. Change in ϕ	yes	no	
c. Additional plant noise	yes	no	no (yes)*
d. Measurement bias	no	yes	yes
e. Change in H	yes	no	
f. Additional measurement noiese	yes	no	no (yes)*
No fault	yes	yes	yes

*In the case of additional noise, either in the state or the measurements, the entries in papentheses denote the steady state

7. CONCLUSION

In this paper the innovations modelling of the Kalman-Bucy discrete filter subject to various sudden changes, was developed. Although $\{y(k)\}$ and $\{\gamma(k)\}$ both contain information of a fault, the use of $\{\gamma(k)\}$ for fault monitoring is fundamentally more attractive in a scheme based on statistical inference.

If no faults occur, the residuals generated by the system measurements can be thought of as sample points from a normal probability distribution with zero mean an variance $C(k,k)$ given by (12-*) If a fault occurs the system output and therefore the residual sequence will

no longer be represented by the well known Kalman-Bucy discrete filter algorithm.
The filter algorithm however, will still operate on the assumed values and as a consequence it will generate residuals which do not belong to the assumed probability distribution.
By obtaining the probability distribution of the residuals generated by the Kalman-Bucy filter after each type of fault, fault detection and isolation can be performed by testing which of the possible probability distributions represent $\{\gamma(k)\}$.
The classification of Table 2 makes the whole scheme appropriate for implementation as an expert system.

APPENDIX I.

Proof of the Theorem: The proof will be by induction. Suppose (13)-(17) hold for time k. At k+1, $\hat{x}(k+1/k+1)$, $\gamma(k+1)$ are calculated by the Kalman filter as,

$$\begin{aligned}\gamma(k+1) &= y(k+1)-H(k+1)\phi(k+1,k)\hat{x}(k/k)\\ &= y_0(k+1)+h_y(k+1,\theta,\Delta P)-H(k+1)\phi(k+1,k)\{\hat{x}_0(k/k)+f(k,\theta,\Delta P)\}\\ &= \gamma_0(k+1)+h_y(k+1,\theta,\Delta P)-H(k+1)\phi(k+1,k)f(k,\theta,\Delta P) \qquad (A.1)\end{aligned}$$

and,

$$\begin{aligned}\hat{x}(k+1/k+1) &= \phi(k+1/k)\hat{x}(k/k)+K(k+1)\gamma(k+1)\\ &= \phi(k+1/k)\{\hat{x}_0(k/k)+f(k,\theta,\Delta P)\}+\\ &\quad +K(k+1)\{\gamma_0(k+1)+h_y(k+1,\theta,\Delta P)-\\ &\quad -H(k+1)\phi(k+1,k)f(k,\theta,\Delta P)\}\\ &= \hat{x}_0(k+1/k+1)+\phi(k+1,k)f(k,\theta,\Delta P)\\ &\quad +K(k+1)\{h_y(k+1,\theta,\Delta P)-H(k+1)\phi(k+1,k)f(k,\theta,\Delta P)\} \qquad (A.2)\end{aligned}$$

where the subscript 0 denotes the value of the parameter that is obtained if no fault occurs. Equations (A.1), (A.2) may be reweitten,

$$\gamma(k+1) = \gamma_0(k+1)+g(k+1,\theta,\Delta P)$$

$$\hat{x}(k+1/k+1) = \hat{x}_0(k+1/k+1)+f(k+1,\theta,\Delta P)$$

where,

$$g(k+1,\theta,\Delta P) = h_y(k+1,\theta,\Delta P)-H(k+1)\phi(k+1,k)f(k,\theta,\Delta P)$$

$$f(k+1,\theta,\Delta P) = \phi(k+1,k)f(k,\theta,\Delta P)+K(k+1)g(k+1,\theta,\Delta P)$$

Therefore, equations (13)-(17) hold fc time k+1.
At $k=\theta$, since the fault has not affected $\hat{x}(\theta-1/\theta-1)$

$$\begin{aligned}\gamma(\theta) &= y(\theta)-H(\theta)\phi(\theta,\theta-1)\hat{x}(\theta-1/\theta-1)\\ &= y_0(\theta)+h_y(\theta,\theta,\Delta P)-H(\theta)\phi(\theta,\theta-1).\hat{x}(\theta-1/\theta-1)\\ &= \gamma_0(\theta)+h_y(\theta,\theta,\Delta P)\end{aligned}$$

and $\hat{x}(\theta/\theta)=\phi(\theta,\theta-1)\hat{x}(\theta-1/\theta-1)+K(\theta)\gamma($

$$= \hat{x}_0(\theta/\theta)+K(\theta)h_y(\theta,\theta,\Delta P)$$

Hence,

$$\gamma(\theta) = \gamma_0(\theta)+g(\theta,\theta,\Delta P)$$

$$\hat{x}(\theta/\theta) = \hat{x}_0(\theta/\theta)+f(\theta,\theta,\Delta P)$$

where,

$$g(\theta,\theta,\Delta P) = h_y(\theta,\theta,\Delta P)$$

$$f(\theta,\theta,\Delta P) = K(\theta)g(\theta,\theta,\Delta P)$$

This completes the proof.

REFERENCES

[1] R. C. Montgomery and A.K. Caglaya "A self-reorganizing digital flig control system for aircraft" presented at the AIAA 12th Aerospace Sciences Meeting, Jan 30-Feb 1,19

[2] P. Sanyal and C. N. Shen, "Bayes cision rule for rapid detection a adaptive estimation scheme with s ce applications" IEEE Trans. Auto mat. Contr. vol. AC-19, pp. 228-2 June 1974.

[3] A.S. Willsky and H. L. Jones, "A neralized likelihood ratio approa to state estimation in linear systems subject to abrupt changes" Proc. 1974 IEEE conf. Decision an Control, Nov. 1974.

[4] A. H. Jazwinski, "Stochastic Processes and Filtering theory" N. Y Academic Press, 1970.

[5] A. Renyi, "Probabillity theory" North Holland, 1970.